Hybrid Performance Bounds for Lie Group and Integer Parameters: Application to GNSS Attitude Estimation

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***Abstract*—The estimation of a mixture of real- and integer-valued parameter vectors appears in different control and signal processing applications. In some particular cases, for instance when dealing with covariance or rotation matrices, the real parameter must be constrained by geometrical properties. This article considers the general estimation problem containing both Lie group and integer parameters, for which the corresponding performance bound is a important missing tool. Leveraging recent results on the Cramér-Rao bound (CRB) for a mixture of real/integer parameters and the CRB for matrix Lie groups, the aim is twofold: i) to derive new performance lower bounds (McAulay-Seidman bound and CRB) on the Lie group-integer space, and ii) to obtain a closed-form CRB expression for multi-antenna Global Navigation Satellite Systems attitude estimation, where observations depend on both an integer carrier phase ambiguity vector and a rotation matrix lying on the Lie group $SO(3)$. The proposed bound is validated through numerical simulations in a realistic scenario to support the discussion.**

***Index Terms*—Lie group, integer space, Cramér-Rao bound, GNSS, precise positioning, attitude estimation.**

I. Introduction

PERFORMANCE lower bounds are extremely useful tools in estimation problems, not only to characterize the asymptotic performance of an estimator, but also for system design [1]–[3]. A plethora of lower bounds in the mean square error (MSE) sense exist in the literature, e.g., deterministic, Bayesian, hybrid, constrained, or misspecified. Within the family of deterministic bounds the most well known is the Cramér-Rao bound (CRB), which i) under certain conditions (i.e., high signal-to-noise ratio (SNR) [4] and/or large number of samples [5]) is asymptotically attained by maximum likelihood estimators (MLEs), and ii) it is the lowest nontrivial bound on the MSE of any unbiased estimator. But the standard CRB has several limitations: it does not predict the threshold phenomena, it may not be tight, and it is not designed to deal with integer-valued parameters or parameters lying in complex/structured spaces such as matrix Lie groups.

Since the seminal CRB, several contributions have proposed solutions to the problems stated above. For example, the Barankin bound (BB) [6] is the tightest (but not computable) deterministic bound, which can be approximated with the McAulay-Seidman bound (MSB) [7]. The MSB has been used to derive a CRB (as a limiting form) able to deal with a hybrid vector containing real and integer-valued parameter vectors [8]. A hybrid bound (containing real and integer parameters) is of interest in a wide variety of control and signal processing problems: 1) in change detection for multivariate time-series modelling [9], where the integer parameter corresponds to the discrete instant at which the model changes, 2) for controlled sensing, where the detection system adaptively adjusts the quality of the information gathered from observations to improve decision-making [10], 3) in image processing, where it is crucial to determine the change on the images treated sequentially by an optical sensor [11], [12], or 4) for Global Navigation Satellite Systems (GNSS) precise positioning techniques exploiting the signal carrier phase, where a set of integer ambiguities must be estimated together with the real-valued platform position/velocity [8], [13].

An additional challenge arises when the unknown real parameter is constrained to respect some geometrical properties. When the parameter lies in a general Riemannian manifold one may resort to the intrinsic CRB (ICRB) [14]–[16], which has also been obtained with some limitations/approximations for the case of matrix Lie groups (LGs), e.g., $SO(3)$ and $SE(3)$ [17]–[20], and is called LG-CRB. These limitations have been recently overcome, leveraging the (intrinsic) MSB, with the exact analytical LG-CRB formula derived in [21]. Nevertheless, in the case where the hybrid estimation problem contains both integer and matrix LG parameters, the corresponding bound does not exist. This may be useful for change detection when the data to process are covariance matrices, as in synthetic-aperture radar [22], or for multi-antenna GNSS-based attitude estimation [23], being the application that ignited this work.

Indeed, attitude estimation ($\mathbf{R} \in SO(3)$), which determines the spatial orientation of an object as the relative orientation between two orthogonal frames, plays a critical role in navigation and control systems. While GNSS is traditionally employed for positioning, its use for attitude determination has gained significant interest for vehicles operating in 3D environments, such as aircraft, spacecraft, and unmanned aerial systems. For instance, the orientation of an aircraft is as critical as its position for applications such as automated landing or flight stabilization. Similarly, satellites and space probes require precise attitude knowledge to meet stringent pointing requirements for communication, imaging, and scientific operations. By configuring multiple GNSS antennas on board, it is possible to estimate the orientation using the precise

This work was partially supported by the DGA/AID projects 2022.65.0082, 2023.65.0083 and 2025.65.0026. S. Labsir is with IPSA/TéSA, France (samy.labsir@ipsa.fr); D. Medina is with DLR, Germany (daniel.ariasmedina@dlr.de); A. Renaux is with University of Paris-Saclay, France (alexandre.renaux@universite-paris-saclay.fr); J. Vilà-Valls and E. Chaumette are with ISAE-SUPAERO, France (jordi.vila-valls@isae-supaero.fr, eric.chaumette@isae-supaero.fr).

measurement of carrier phase differences between antennas [23], [24]. To achieve 3D orientation estimates, a minimum of three non-coplanar antennas must be deployed, with their relative positions accurately surveyed in the vehicle's body frame. The achievable precision is intrinsically linked to the accuracy of inter-antenna baseline measurements, which in turn depends on the quality of the carrier phase observations, and is inversely proportional to the antenna separation. A performance bound for such estimation problem, to verify the efficiency of standard GNSS attitude estimators and assess the system limitations, is an fundamental missing tool.

The main goal of this article is to derive a general hybrid CRB (called LG-IS-CRB) for the joint estimation of a LG parameter (e.g., rotation matrix) and an integer-valued parameter vector, which does not exist in the literature and is the natural extension of the recent results in [8], [21]. We leverage the space $G \times \mathbb{Z}$ (LG $\times$ integers) that can be equipped with a group structure, in which we can define algebraic operations. Then, the Barankin condition already defined on LGs [21] can be generalized to $G \times \mathbb{Z}$, which is used to derive the new MSB and LG-IS-CRB. Finally, we obtain a closed-form LG-IS-CRB expression for the multi-antenna GNSS-based attitude estimation problem [25], which is validated through numerical simulations in a realistic scenario.

II. Background on Lie groups and LG-CRB

A. Lie group: definition

A matrix LG G is a matrix space equipped with the structures of both a smooth manifold and a group. As a smooth manifold, G allows for the definition of integration and differentiation operations. In particular, this structure enables the specification of a tangent space at each element of G. The tangent space at the identity element of G, is called the Lie algebra, itself in bijection with $\mathbb{R}^m$. Each LG element that is sufficiently close to the neutral element can be associated with an element of $\mathbb{R}^m$ through the logarithm and exponential maps, defined as $\text{Exp}_G^\wedge(.) \; : \mathbb{R}^m \to G$ and $\text{Log}_G^\vee(.) \; : G \to \mathbb{R}^m$. More details can be found in [21].

B. McAulay-Seidman and Cramér-Rao bounds on Lie groups

Let us consider two LGs G (with dimension P) and G'. The observations $\mathbf{Y} \in G'$ are connected to an unknown parameter $\mathbf{R}_0 \in G$ through the likelihood $p(\mathbf{Y}|\mathbf{R}_0)$ and we seek for an estimator $\widehat{\mathbf{R}}_0$. Using the intrinsic bias definition provided in [21], and given by

$$\mathbf{b}_{\mathbf{Y}|\mathbf{R}}(\mathbf{R}_0, \widehat{\mathbf{R}}_0) \triangleq \mathbb{E}_{p(\mathbf{Y}|\mathbf{R})}\left(\text{Log}_G^\vee\left(\mathbf{R}_0^{-1}\widehat{\mathbf{R}}_0\right)\right) \forall \mathbf{R} \in G, \quad (1)$$

a condition of intrinsic uniform unbiasedness for the LG estimator $\widehat{\mathbf{R}}_0$ is

$$\mathbf{b}_{\mathbf{Y}|\mathbf{R}}\left(\mathbf{R}_0, \widehat{\mathbf{R}}_0\right) = \text{Log}_G^\vee\left(\mathbf{R}_0^{-1}\mathbf{R}\right) \;, \forall \mathbf{R} \in G, \quad (2)$$

leading to the Barankin bound on LGs [21]. This bound cannot be calculated analytically, but it can approximated using a set of test points, resulting in the MSB on LGs.

Definition II-B.1 (McAulay-Seidman bound on LGs). Let a set of test points $\mathbf{R}^{(1:L)} = \{\mathbf{R}^{(1)}, \ldots, \mathbf{R}^{(L)}\}$. The intrinsic MSE (IMSE) is bounded by the intrinsic McAulay-Seidman bound (LG-MSB) [21]

$$\mathbb{E}\left(\text{Log}_G^\vee\left(\mathbf{R}_0^{-1}\widehat{\mathbf{R}}_0\right) \text{Log}_G^\vee\left(\mathbf{R}_0^{-1}\widehat{\mathbf{R}}_0\right)^\top\right) \succeq \mathbf{P}_{\text{LG-MSB}}, \quad (3)$$

where $\mathbb{E}(.) \triangleq \mathbb{E}_{p(\mathbf{Y}|\mathbf{R}_0)}(.)$ and

$$\begin{aligned} \mathbf{P}_{\text{LG-MSB}} &= \boldsymbol{\Delta}_G \mathcal{I}_{\text{LG-MSB}}^{-1} \boldsymbol{\Delta}_G^\top, \quad (4)\\ \mathcal{I}_{\text{LG-MSB}} &= \mathbb{E}\left(\boldsymbol{v}_{\mathbf{R}_0}\left(\mathbf{Y}; \mathbf{R}^{(1:L)}\right) \boldsymbol{v}_{\mathbf{R}_0}\left(\mathbf{Y}; \mathbf{R}^{(1:L)}\right)^\top\right),\\ \left[\boldsymbol{v}_{\mathbf{R}_0}\left(\mathbf{Y}; \mathbf{R}^{(1:L)}\right)\right]_l &= \frac{p(\mathbf{Y}|\mathbf{R}^{(l)})}{p(\mathbf{Y}|\mathbf{R}_0)},\\ \boldsymbol{\Delta}_G &= \left[\text{Log}_G^\vee\left(\mathbf{R}_0^{-1}\mathbf{R}^{(1)}\right), \ldots, \text{Log}_G^\vee\left(\mathbf{R}_0^{-1}\mathbf{R}^{(L)}\right)\right]. \end{aligned}$$

Definition II-B.2 (Cramér-Rao bound on LGs). Let us consider the following set of $L = P + 1$ test points:

$$\mathbf{R}^{(1:L)} = \left\{\mathbf{R}_0, \mathbf{R}_0 \text{Exp}_G^\wedge(\mathbf{i}_1\delta_1), \ldots, \mathbf{R}_0 \text{Exp}_G^\wedge(\mathbf{i}_{L-1}\delta_{L-1})\right\}$$

with $\forall l \in \{1, \ldots, L-1\}$ $\mathbf{i}_l^\top = \left[0, \ldots, \underbrace{1}_{\text{component } l}, \ldots, 0\right]$.

When $\delta_l \to 0$, the inequality (3) is still valid and we obtain the following expression for the LG-CRB [21]:

$$\begin{aligned} \mathbf{P}_{\text{LG-CRB}}(\mathbf{R}_0) &= \mathcal{I}_{\text{LG-CRB}}^{-1}, \quad (5)\\ \mathcal{I}_{\text{LG-CRB}} &= \mathbb{E}\left(\mathbf{s}(\mathbf{Y}, \mathbf{R}_0)\, \mathbf{s}(\mathbf{Y}, \mathbf{R}_0)^\top\right), \end{aligned}$$

where $\mathbf{s}(\mathbf{Y}, \mathbf{R}_0) = \left.\frac{\partial \log p(\mathbf{Y}|\mathbf{R}_0 \text{Exp}_G^\wedge(\boldsymbol{\epsilon}))}{\partial \boldsymbol{\epsilon}}\right|_{\boldsymbol{\epsilon}=\mathbf{0}} \forall \boldsymbol{\epsilon} \in \mathbb{R}^P$

III. Development of the CRB on LG-Integer Space

In this section, we adapt the LG-CRB to a hybrid parameter vector containing both LG and integer-valued elements. Then, the aim is to generalize the CRBs proposed in [8] and [21]. To achieve that, we define a new group structure on which we can specify test points similar to Def. II-B.1. Then, new hybrid MSB and CRB on this LG-integer space are derived.

A. Definition of Lie group-integer space (LG × IS)

Let us consider a matrix LG $G \subset \mathbb{R}^{n\times n}$ of dimension P and the set of integers $\mathbb{Z}^K$. We define the product set $G \times \mathbb{Z}^K$ such that $\forall \mathbf{M} \in G \times \mathbb{Z}^K$,

$$\mathbf{M} = \begin{bmatrix} \mathbf{R} & \mathbf{0}_{n\times K} & \mathbf{0}_{n\times 1}\\ \mathbf{0}_{K\times n} & \mathbf{I}_{K\times K} & \mathbf{z}\\ \mathbf{0}_{1\times n} & \mathbf{0}_{1\times K} & 1 \end{bmatrix} \text{ with } \mathbf{R} \in G \text{ and } \mathbf{z} \in \mathbb{Z}^K. \quad (6)$$

One can see that $G \times \mathbb{Z}^K$ defines a group with an internal law of matrix multiplication, ensuring that the subset maintains its classical plus operation. Since G is a LG, we can generalize the logarithmic map to non-differential map to the group $G \times \mathbb{Z}^K$, $\forall \mathbf{M}$ written in the form (6), as follows

$$\boldsymbol{l}_{G\times\mathbb{Z}^K}(\mathbf{M}) \triangleq \begin{bmatrix} \text{Log}_G^\vee(\mathbf{R})\\ \mathbf{z} \end{bmatrix}, \quad (7)$$

and the exponential map $\forall (\boldsymbol{\delta}_R, \boldsymbol{\delta}_z) \in \mathbb{R}^P \times \mathbb{Z}^K$ as:

$$\mathbf{E}_{G\times\mathbb{Z}^K}([\boldsymbol{\delta}_R, \boldsymbol{\delta}_z]^\top) \triangleq \begin{bmatrix} \text{Exp}_G^\wedge(\boldsymbol{\delta}_R) & \mathbf{0}_{n\times K} & \mathbf{0}_{n\times 1}\\ \mathbf{0}_{K\times n} & \mathbf{I}_{K\times K} & \boldsymbol{\delta}_z\\ \mathbf{0}_{1\times n} & \mathbf{0}_{1\times K} & 1 \end{bmatrix}. \quad (8)$$

Note that $G \times \mathbb{Z}^K$ can also be endowed with a Lie group structure by considering it as a countably infinite non-connected copies of G. Although this structure is mathematically interesting, it poses significant practical challenges and substantially limits the applicability of differential tools that are foundational in many estimation frameworks. Also, hybrid bounds are typically derived on product spaces such as $\mathbb{R}\times\mathbb{Z}$, where the continuous and discrete parameters are treated independently [26]–[28]. In our setting, where the continuous component evolves on a Lie group G, it is natural to extend this framework by considering the product space $G \times \mathbb{Z}^K$. This generalization not only respects the underlying geometric structure but also facilitates meaningful comparisons between our proposed bounds and existing estimation algorithms.

B. McAulay-Seidman and Cramér-Rao bounds on LG × IS

We consider a set of random observations $\mathbf{Y} \in G'$ depending on both LG parameter $\mathbf{R}_0 \in G$ and integer parameter $\mathbf{z}_0 \in \mathbb{Z}^K$ through the likelihood $p(\mathbf{Y}|\mathbf{M}_0)$, with:

$$\mathbf{M}_0 = \begin{bmatrix} \mathbf{R}_0 & \mathbf{0}_{n\times K} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{K\times n} & \mathbf{I}_{K\times K} & \mathbf{z}_0 \\ \mathbf{0}_{1\times n} & \mathbf{0}_{1\times K} & 1 \end{bmatrix} \in G \times \mathbb{Z}^K.$$

In the following, we provide two theorems: the first one gives a hybrid MSB on LG × IS, and the second one provides a generalization of a hybrid CRB on this same space.

Theorem 1 (Hybrid McAulay-Seidman bound on LG × IS). *Let $\mathbf{Y}$ a set of LG observations on a LG G', and an unknown parameter $\mathbf{M}_0 \in G \times \mathbb{Z}^K$. If the following unbiasedness on $\widehat{\mathbf{M}}_0$ holds*

$$\mathbf{b}_{\mathbf{Y}|\mathbf{M}}\left(\mathbf{M}_0, \widehat{\mathbf{M}}_0\right) = \boldsymbol{l}_{G\times\mathbb{Z}^K}\left(\mathbf{M}_0^{-1}\mathbf{M}\right) \forall\, \mathbf{M} \in G \times \mathbb{Z}^K, \quad (9)$$

for a set of test points $\mathbf{M}^{(1:L)}$, divided into test points $\mathbf{M}_{LG}^{(1:L_1)}$ on G, and test points on $\mathbb{Z}^K$, $\mathbf{M}_{IS}^{(1:L-L_1)}$: $\mathbf{M}^{(1:L)} = \left[\mathbf{M}_{LG}^{(1:L_1)}\, \mathbf{M}_{IS}^{(1:L-L_1)}\right]$, then the hybrid MSB on LG × IS, $\mathbf{P}_{\text{LG-IS-MSB}}$, verifies

$$\mathbb{E}\left(\boldsymbol{l}_{G\times\mathbb{Z}^K}\left(\mathbf{M}_0^{-1}\widehat{\mathbf{M}}_0\right) \boldsymbol{l}_{G\times\mathbb{Z}^K}\left(\mathbf{M}_0^{-1}\widehat{\mathbf{M}}_0\right)^\top\right) \succeq \mathbf{P}_{\text{LG-IS-MSB}} \triangleq \boldsymbol{\Delta}_{\text{LG-IS}}\, \mathcal{I}_{\text{LG-IS-MSB}}^{-1} \boldsymbol{\Delta}_{\text{LG-IS}}^\top, \quad (10)$$

where

$$\mathcal{I}_{\text{LG-IS-MSB}} = \mathbb{E}\left(\mathbf{v}_{\mathbf{M}_0}(\mathbf{Y};\{\mathbf{M}^{(1:L)}\})\mathbf{v}_{\mathbf{M}_0}(\mathbf{Y};\{\mathbf{M}^{(1:L)}\})^\top\right),$$

$$\left[\mathbf{v}_{\mathbf{M}_0}(\mathbf{Y};\mathbf{M}^{(1:L)})\right]_l = \frac{p(\mathbf{Y}|\mathbf{M}^{(l)})}{p(\mathbf{Y}|\mathbf{M}_0)} \;\; \forall l \in \{1,\dots,L\},$$

$$[\boldsymbol{\Delta}_{\text{LG-IS}}]_l = \boldsymbol{l}_{G\times\mathbb{Z}^K}\left(\mathbf{M}_0^{-1}\mathbf{M}_{LG}^{(l)}\right) \forall l \in \{1,\dots,L_1\},$$

$$[\boldsymbol{\Delta}_{\text{LG-IS}}]_l = \boldsymbol{l}_{G\times\mathbb{Z}^K}\left(\mathbf{M}_0^{-1}\mathbf{M}_{IS}^{(l-L_1)}\right) \forall l \in \{L_1+1,\dots,L\}.$$

Proof. See Appendix A-A. □

Remark 1. The term $\mathcal{I}_{\text{LG-IS-MSB}}$ can be interpreted as a Fisher information matrix quantifying the information brought by each test point on the model. Regarding the term $\boldsymbol{\Delta}_{\text{LG-IS}}$, it quantifies the similarity between the true parameter $\mathbf{M}_0$ with each test point.

Theorem 2 (Hybrid Cramér-Rao bound on LG × IS). *Let P be the dimension of G. Let us consider $L_1 = P + 1$ test points on G, generated as in Def. II-B.2:*

$$\mathbf{M}_{LG}^{(1:L_1)} = \{\mathbf{M}_0, \mathbf{M}_0\, \mathbf{E}_{G\times\mathbb{Z}^K}[\mathbf{i}_1\, \delta_1, \mathbf{0}]^\top, \dots, \mathbf{M}_0\, \mathbf{E}_{G\times\mathbb{Z}^K}[\mathbf{i}_P\, \delta_P, \mathbf{0}]^\top\}. \quad (11)$$

Following the approach in [8], let us consider $2\,K$ test points

$$\mathbf{M}_{IS}^{(1:2K)} = \{\mathbf{M}_0\, \mathbf{E}_{G\times\mathbb{Z}^K}\left[\mathbf{0}, \mathbf{z}^{(1)}\right]^\top, \dots, \mathbf{M}_0\, \mathbf{E}_{G\times\mathbb{Z}^K}\left[\mathbf{0}, \mathbf{z}^{(2K)}\right]^\top\}, \quad (12)$$

with $\mathbf{z}^{(i)} = \mathbf{z}_0 + \boldsymbol{\gamma}^{(i)} \forall i \in \{1,\dots,2K\}$ and $\boldsymbol{\gamma}^{(i)} = (-1)^{(i-1)}\mathbf{i}_{P+\left[\frac{i+1}{2}\right]}$. Taking advantage of (10), the hybrid CRB on LG × IS, $\mathbf{P}_{\text{LG-IS-CRB}}$, verifies

$$\mathbb{E}\left(\boldsymbol{l}_{G\times\mathbb{Z}^K}\left(\mathbf{M}_0^{-1}\widehat{\mathbf{M}}_0\right) \boldsymbol{l}_{G\times\mathbb{Z}^K}\left(\mathbf{M}_0^{-1}\widehat{\mathbf{M}}_0\right)^\top\right) \succeq \mathbf{P}_{\text{LG-IS-CRB}} \triangleq \tilde{\boldsymbol{\Delta}}(\{\boldsymbol{\gamma}^{(i)}\}_{i=1}^{2K})\, \mathcal{I}_{\text{LG-IS-CRB}}^{-1}\, \tilde{\boldsymbol{\Delta}}(\{\boldsymbol{\gamma}^{(i)}\}_{i=1}^{2K})^\top, \quad (13)$$

where

$$\tilde{\boldsymbol{\Delta}}(\{\boldsymbol{\gamma}^{(i)}\}_{i=1}^{2K}), \triangleq \begin{bmatrix} \mathbf{i}_1 & \cdots & \mathbf{i}_P & 0 & \cdots & 0 \\ \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{\gamma}^{(1)} & \cdots & \boldsymbol{\gamma}^{(2K)} \end{bmatrix} \quad (14)$$

and

$$\mathcal{I}_{\text{LG-IS-CRB}} = \begin{bmatrix} \mathcal{I}_{\mathbf{R}_0} & \mathcal{I}_{\mathbf{R}_0,\mathbf{z}_0} \\ \mathcal{I}_{\mathbf{R}_0,\mathbf{z}_0}^\top & \mathcal{I}_{\mathbf{z}_0} \end{bmatrix}. \quad (15)$$

Considering the vector $\left[\boldsymbol{\epsilon}, \boldsymbol{\gamma}^{(j)}\right]^\top \forall \boldsymbol{\epsilon} = \left[\epsilon_1, \dots, \epsilon_P\right]^\top \in \mathbb{R}^P$ and introducing the notation

$$\tilde{p}(\mathbf{Y};\mathbf{M}_0,\boldsymbol{\epsilon}) \triangleq p(\mathbf{Y}|\mathbf{M}_0\, \mathbf{E}_{G\times\mathbb{Z}^K}[\boldsymbol{\epsilon},\mathbf{0}]^\top,$$
$$\tilde{p}(\mathbf{Y};\mathbf{M}_0,\boldsymbol{\gamma}^{(j)}) \triangleq p(\mathbf{Y}|\mathbf{M}_0\, \mathbf{E}_{G\times\mathbb{Z}^K}[\mathbf{0},\boldsymbol{\gamma}^{(j)}]^\top,$$

the different terms in $\mathcal{I}_{\text{LG-IS-CRB}}$ are

$$\mathcal{I}_{\mathbf{R}_0} = \mathbb{E}\left(\frac{\partial \log \tilde{p}(\mathbf{Y};\mathbf{M}_0,\boldsymbol{\epsilon})}{\partial \boldsymbol{\epsilon}} \left.\frac{\partial \log \tilde{p}(\mathbf{Y};\mathbf{M}_0,\boldsymbol{\epsilon}))}{\partial \boldsymbol{\epsilon}}^\top\right|_{\boldsymbol{\epsilon}=\mathbf{0}}\right), \quad (16)$$

$$(\mathcal{I}_{\mathbf{R}_0,\mathbf{z}_0})_{i,j} = \mathbb{E}\left(\left.\frac{\partial \log \tilde{p}(\mathbf{Y};\mathbf{M}_0,\boldsymbol{\epsilon}))}{\partial \epsilon_i}\right|_{\epsilon_i=0} \frac{\tilde{p}(\mathbf{Y};\mathbf{M}_0,\boldsymbol{\gamma}^{(j)}) - p(\mathbf{Y}|\mathbf{M}_0)}{p(\mathbf{Y}|\mathbf{M}_0)}\right) \quad (17)$$

$$\mathcal{I}_{\mathbf{z}_0} = \mathbb{E}\left(\mathbf{c}\mathbf{c}^\top\right) - \mathbf{1}_{2K}\mathbf{1}_{2K}^\top, \quad (18)$$

$$[\mathbf{c}]_i = \frac{\tilde{p}(\mathbf{Y};\mathbf{M}_0,\boldsymbol{\gamma}^{(i)})}{p(\mathbf{Y}|\mathbf{M}_0)} \;\; \forall i \in \{1,\dots,2K\}, \quad (19)$$

with $\mathbf{1}_{2K}$ a vector of 1 with size $2K$.

Proof. See Appendix A-B. □

Remark 2. The Fisher information matrix $\mathcal{I}_{\text{LG-IS-CRB}}$ encompasses the information related to the LG parameter via the term $\mathcal{I}_{\boldsymbol{R}_0}$, and to the discrete parameter through $\mathcal{I}_{\boldsymbol{z}_0}$. Furthermore, it enables the computation of the LG–discrete cross-information term, denoted by $\mathcal{I}_{\boldsymbol{R}_0,\boldsymbol{z}_0}$.

Remark 3. Although this approach is not directly relevant to our work, one could, in principle, derive the bound by equipping the space with the Lie group structure previously mentioned and applying the result II-B-2. However, a major shortcoming of this construction is that, due to the lack of connectedness, the resulting bound cannot be defined globally across the group. Moreover, this structure does not allow for the explicit representation of cross-correlation terms between the discrete and continuous components, that is fundamental for comparisons with hybrid estimation algorithms.

Remark 4. As established by the condition (9), the proposed LG-IS-CRB is only valid for estimators that are unbiased in the LG sense. In our simulations, we will observe that this condition is clearly satisfied.

IV. Closed-Form LG-IS-CRB Expression for Multi-Antenna GNSS Attitude Estimation

A. Problem formulation and system model

Consider a vehicle equipped with a system of $N+1$ GNSS antennas and receivers. One of these antennas is denoted as main (m) and the other N as secondary. The local or body frame ($\mathcal{B}$) of the vehicle is surveyed, i.e., the antenna positions (${}_{\mathcal{B}}\mathbf{p}_m$, ${}_{\mathcal{B}}\mathbf{p}_j, j=1,\dots,N$) are accurately known. The baseline vector between the main and the jth secondary antenna is given by ${}_{\mathcal{B}}\mathbf{b}_{j,m} = {}_{\mathcal{B}}\mathbf{p}_j - {}_{\mathcal{B}}\mathbf{p}_m,$, and the N combinations of secondary and main antennas is expressed as the vector

$$\mathbf{b}^\top = \left[{}_{\mathcal{B}}\mathbf{b}_{1,m}^\top, \dots, {}_{\mathcal{B}}\mathbf{b}_{N,m}^\top\right] . \tag{20}$$

For each of the antennas, $n+1$ satellites are tracked by the receivers and produce code and carrier phase pseudorange measurements.

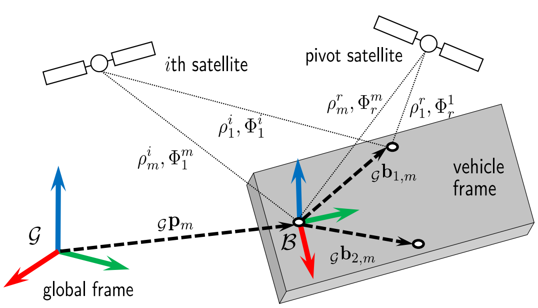

Fig. 1. Scheme for the collection of GNSS data over different antennas.

The models for the carrier phase (Φ_j^i) and code (ρ_j^i) measurements for the ith satellite at the jth antenna are

$$\begin{aligned}\rho_j^i &= \|{}_{\mathcal{G}}\mathbf{p}^i - {}_{\mathcal{G}}\mathbf{p}_j\| + I^i + T^i + cdt_j - cdt^i + \varepsilon_j^i, \\ \Phi_j^i &= \|{}_{\mathcal{G}}\mathbf{p}^i - {}_{\mathcal{G}}\mathbf{p}_j\| - I^i + T^i + cdt_j - cdt^i + \lambda z_j^i + \epsilon_j^i.\end{aligned} \tag{21}$$

We use superscripts and right subscripts to denote satellites and antennas, respectively, while left subscripts indicate the reference frame. GNSS pseudoranges capture satellite-to-antenna distances, atmospheric delays (I^i, T^i), clock offsets (cdt^i, cdt_j), and noise (ε_j^i, ϵ_j^i). Carrier phase measurements also include integer ambiguities (z_j^i) scaled by the wavelength (λ). Fig. 1 summarizes the notation and system. The GNSS attitude model relies on double-difference (DD) combinations to cancel nuisance terms and extract directional information, by comparing signals from secondary antennas to the primary one and a pivot satellite. DD code and phase measurements between the main and jth antenna, and the reference and ith satellite, are defined as follows:

$$\begin{aligned}DD\Phi_{j,m}^{i,r} &\triangleq \Phi_j^i - \Phi_m^i - \left(\Phi_j^r - \Phi_m^r\right), \\ DD\rho_{j,m}^{i,r} &\triangleq \rho_j^i - \rho_m^i - \left(\rho_j^r - \rho_m^r\right),\end{aligned} \tag{22}$$

whose linearized observation model is given by

$$DD\Phi_{j,m}^{i,r} = -(\mathbf{u}^i - \mathbf{u}^r)^\top \mathbf{R}\ {}_{\mathcal{B}}\mathbf{b}_{j,m} + \lambda z_{j,m}^{i,r}, \tag{23}$$

$$DD\rho_{j,m}^{i,r} = -(\mathbf{u}^i - \mathbf{u}^r)^\top \mathbf{R}\ {}_{\mathcal{B}}\mathbf{b}_{j,m}, \tag{24}$$

where $\mathbf{u}^i$ is the unitary pointing vector between the ith satellite and the vehicle, and $z_{j,m}^{i,r}$ is the integer ambiguity for the DD carrier phase. The overall set of measurements ($\mathbf{y} \in \mathbb{R}^{2\cdot n\cdot N}$) gathers the vectors of DD observations between the main and each secondary antenna ($\mathbf{y}_{j,m} \in \mathbb{R}^{2n}$), such that

$$\mathbf{y} = \operatorname{vec}\left(\left[\mathbf{y}_{1,m}, \dots, \mathbf{y}_{N,m}\right]\right), \tag{25}$$

$$\mathbf{y}_{j,m}^\top = \left[DD\Phi_{j,m}^{1,r}, \dots, DD\Phi_{j,m}^{n,r}, DD\rho_{j,m}^{1,r}, \dots, DD\rho_{j,m}^{n,r}\right].$$

Definition 1 (Attitude Mixed Model). Let $\mathbf{h}(\cdot)$ be an observation function and $\mathbf{\Sigma} \in \mathbb{R}^{2\cdot n\cdot N, 2\cdot n\cdot N}$ the covariance matrix for the observations. The attitude mixed model is written as

$$\mathbf{y} \sim \mathcal{N}\left(\mathbf{h}\left(\mathbf{R}_0, \mathbf{z}_0\right), \mathbf{\Sigma}\right), \tag{26}$$

$$\mathbf{z}_0 \in \mathbb{Z}^{n\cdot N}, \mathbf{R}_0 \in SO(3) \ \ \text{(unknown parameters)} \tag{27}$$

with $\mathbf{z}_0$ the vector of integer ambiguities, $\mathbf{R}_0$ the rotation matrix between body and global frames, $\mathbf{h}(\cdot)$ given by

$$\mathbf{h}(\mathbf{R}_0, \mathbf{z}_0) = \mathbf{A}\mathbf{z}_0 + \left(\begin{bmatrix}\mathbf{I}_N \\ \mathbf{I}_N\end{bmatrix} \otimes (\mathbf{C}\mathbf{R}_0)\right)\mathbf{b}, \tag{28}$$

and $\mathbf{A} \in \mathbb{R}^{2\cdot n\cdot N, n\cdot N}$, $\mathbf{C} \in \mathbb{R}^{2\cdot n, 3}$ and $\mathbf{b}$ defined as

$$\mathbf{A} = \begin{bmatrix}\lambda \mathbf{I}_{n\cdot N} \\ \mathbf{0}_{n\cdot N}\end{bmatrix}, \mathbf{C} = \begin{bmatrix}-\left(\mathbf{u}^1 - \mathbf{u}^r\right)^\top \\ \vdots \\ -\left(\mathbf{u}^n - \mathbf{u}^r\right)^\top\end{bmatrix}, \mathbf{b} = \begin{bmatrix}{}_{\mathcal{B}}\mathbf{b}_{1,m}, \\ \vdots \\ {}_{\mathcal{B}}\mathbf{b}_{N,m}\end{bmatrix}. \tag{29}$$

The observations' covariance matrix $\mathbf{\Sigma}$ can be modeled as described in [29, Ch. 3].

B. Expression of the LG-IS-CRB for GNSS attitude estimation

Theorem 3 (LG-IS-CRB for the GNSS attitude model). *Let us consider the unknown LG-integer parameter*

$$\mathbf{M}_0 = \begin{bmatrix}\mathbf{R}_0 & \mathbf{0}_{3\times K} & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{K\times 3} & \mathbf{I}_{K\times K} & \mathbf{z}_0 \\ \mathbf{0}_{1\times 3} & \mathbf{0}_{1\times K} & 1\end{bmatrix} \in G \times \mathbb{Z}^K.$$

Knowing the observations $\mathbf{y}$, distributed according to (26), and a set of discrete test points $\{\mathbf{z}^{(k)}\}_{k=1}^{2K}$, we can compute a closed-form of $\mathbf{P}_{\text{LG-IS-CRB}}$ (13). The matrix terms of $\mathcal{I}_{\text{LG-IS-CRB}}$ (15), $\forall (i,j) \in [\![1:3]\!]^2$, $\forall (l,k) \in [\![1:2K]\!]^2$ are:

$$\begin{aligned}(\mathcal{I}_{\mathbf{R}_0})_{i,j} = &\left(\left(\begin{bmatrix}\mathbf{I}_N \\ \mathbf{I}_N\end{bmatrix} \otimes \mathbf{C}\,\mathbf{R}_0\,\mathbf{G}_i\right)\mathbf{b}\right)^\top \\ &\times \mathbf{\Sigma}^{-1}\left(\begin{bmatrix}\mathbf{I}_N \\ \mathbf{I}_N\end{bmatrix} \otimes (\mathbf{C}\,\mathbf{R}_0\,\mathbf{G}_j)\,\mathbf{b}\right),\end{aligned} \tag{30}$$

$$(\mathcal{I}_{\mathbf{R}_0,\mathbf{z}_0})_{i,k} = \left(\left(\begin{bmatrix}\mathbf{I}_N \\ \mathbf{I}_N\end{bmatrix} \otimes \mathbf{C}\,\mathbf{R}_0\,\mathbf{G}_i\right)\mathbf{b}\right)^\top \mathbf{\Sigma}^{-1}\mathbf{A}\boldsymbol{\gamma}^{(k)}, \tag{31}$$

$$(\mathcal{I}_{\mathbf{z}_0})_{l,k} = \exp\left(0.5\ \left(\mathbf{m}_{l,k}^\top \mathbf{\Sigma}\,\mathbf{m}_{l,k} - \delta_{l,k}\right)\right) - 1, \tag{32}$$

where $\{\mathbf{G}_i\}_{i=1}^3$ *is a basis of the Lie algebra of* $SO(3)$*, denoted* $\mathfrak{so}(3)$ *which is the set of skew symmetric matrices of size* 3×3*. Furthermore:*

$$\mathbf{m}_{l,k} = \mathbf{\Sigma}^{-1}\mathbf{h}(\mathbf{R}_0,\mathbf{z}^{(l)}) + \mathbf{\Sigma}^{-1}\mathbf{h}(\mathbf{R}_0,\mathbf{z}^{(k)}) - \mathbf{\Sigma}^{-1}\,\mathbf{h}(\mathbf{R}_0,\mathbf{z}_0) \tag{33}$$

$$\delta_{l,k} = \mathbf{h}(\mathbf{R}_0,\mathbf{z}^{(l)})^\top\mathbf{\Sigma}^{-1}\mathbf{h}(\mathbf{R}_0,\mathbf{z}^{(l)}) + \mathbf{h}(\mathbf{R}_0,\mathbf{z}^{(k)})^\top \times \mathbf{\Sigma}^{-1}\,\mathbf{h}(\mathbf{R}_0,\mathbf{z}^{(k)}) - \mathbf{h}(\mathbf{R}_0,\mathbf{z}_0)^\top\mathbf{\Sigma}^{-1}\,\mathbf{h}(\mathbf{R}_0,\mathbf{z}_0). \tag{34}$$

Proof. See Appendix A-C. □

V. Validation

A. Derivation of the estimator

The observation model in (26) leads to an optimization problem with unknown LG and integer parameters. From a MLE perspective, the estimates $\hat{\mathbf{R}}_0, \hat{\mathbf{z}}_0$ can be found as

$$(\hat{\mathbf{R}}_0,\hat{\mathbf{z}}_0) = \arg\min_{\substack{\mathbf{z}_0\in\mathbb{Z}^{n\cdot N}\\ \mathbf{R}_0\in SO(3)}} \|\mathbf{y}-\mathbf{h}(\mathbf{R}_0,\mathbf{z}_0)\|^2_{\mathbf{\Sigma}}. \tag{35}$$

A closed-form estimator for (35) is not available, due to the integer nature of vector $\mathbf{z}_0$. Instead, one typically applies an orthogonal decomposition and expresses the quadratic optimization problem in (35) as the sum of three squares [30],

$$\min_{\substack{\mathbf{z}_0\in\mathbb{Z}^{n\cdot N}\\ \mathbf{R}_0\in SO(3)}} \|\mathbf{y}-\mathbf{h}(\mathbf{R}_0,\mathbf{z}_0)\|^2_{\mathbf{\Sigma}} = \|\check{\mathbf{e}}\|^2_{\mathbf{\Sigma}} + \tag{36}$$
$$\min_{\mathbf{z}_0\in\mathbb{Z}^{n\cdot N}}\left(\|\check{\mathbf{z}}_0-\mathbf{z}_0\|^2_{\mathbf{P}_{\check{\mathbf{z}}_0}} + \min_{\mathbf{R}\in SO(3)}\left\|\check{\mathbf{R}}(\mathbf{z}_0)-\mathbf{R}\right\|^2_{\mathbf{P}_{\check{\mathbf{R}}(\mathbf{z}_0)}}\right),$$

with $\check{\mathbf{e}} = \mathbf{y}-\mathbf{h}(\check{\mathbf{R}}_0,\check{\mathbf{z}}_0)$ a vector of residuals based on a set of auxiliary variables, commonly referred to as *float solution*, which disregards the integer-ness of $\mathbf{z}_0$, such that

$$(\check{\mathbf{R}}_0,\check{\mathbf{z}}_0) = \arg\min_{\substack{\mathbf{z}_0\in\mathbb{R}^{n\cdot N}\\ \mathbf{R}_0\in SO(3)}} \|\mathbf{y}-\mathbf{h}(\mathbf{R}_0,\mathbf{z}_0)\|^2_{\mathbf{\Sigma}}. \tag{37}$$

The *fixed solution* is obtained in a second step by jointly estimating the integer ambiguities and the rotation between the two frames of interest. Available estimators can be divided into two families: i) *baseline methods* estimate first the inter-antenna baselines in the global frame, and then the rotation between frames from a Wahba's problem, and ii) *direct methods*, directly estimate the rotation between frames, with algorithms varying on the attitude parametrization and on the use of optimization for manifolds. Examples of *baseline* estimators are the baseline LAMBDA [13] and the baseline-length constrained C-LAMBDA [31], while *direct* estimators include MC-LAMBDA [32], RieMOCAD [33] and Q-LAMBDA [29].

B. Experimental setup and results

We validate the proposed LG-IS-CRB on a simulated GNSS attitude problem. A vehicle carries 4 antennas spaced $b = 10$ m, with baselines $\mathbf{b}^\top = b\cdot[\,[1,0,0],[0,1,0],[0.57,0.57,0.57]\,]$, each tracking $n = 11$ satellites (GPS L1, San Fernando IGS, 2024/12/03 10:00). The noise of code observations ($\varepsilon \sim \mathcal{N}(0,\sigma_\rho^2)$ in Eq. (21)) is swept over a wide range, and carrier phase noise is two order of magnitude smaller. The performance of three estimators, baseline LAMBDA, Q-LAMBDA and MC-LAMBDA, is evaluated in terms of intrinsic MSE (IMSE) on $N_r = 1000$ estimator realizations $\{\widehat{\mathbf{R}}_0^{(nr)}\}_{nr=1}^{N_r}$

$$\text{IMSE} \simeq \frac{1}{N_r}\sum_{nr=1}^{N_r}\|\text{Log}^\vee_{SO(3)}\left(\mathbf{R}_0^{-1}\widehat{\mathbf{R}}_0^{(nr)}\right)\|^2, \tag{38}$$

and checked against the derived LG-IS-CRB. For each estimator two types of IMSE results are shown in Fig. 2: *float* refer to $\check{\mathbf{R}}_0$ in (37), and *fix* to $\hat{\mathbf{R}}_0$ in (35).

First, we notice the precision gain when the integer constraint on the carrier phase ambiguities is preserved, manifest as the difference between the LG-IS-CRB and the LG × real space CRB (LG-RS-CRB). As the product of a LG with a real space is also a LG, the associated LG-RS-CRB was already derived in [21]. Secondly, we observe the performance of the estimators with respect to the bounds, where we can identify three performance regions:

a) Low noise regime ($\sigma_\rho \leq 0.1$ m): all estimators correctly estimate the complete vector of integer ambiguities, with Q-LAMBDA and MC-LAMBDA attaining the LG-IS-CRB and showcasing asymptotic efficiency, and LAMBDA being a suboptimal estimator. This result validates the proposed LG-IS-CRB. The same behavior is shown by the *float* solution and the LG-RS-CRB, where LAMBDA is not efficient.

b) Threshold region ($0.1 \leq \sigma_\rho \leq 10$ m): both LAMBDA and Q-LAMBDA correctly estimate *only some* ambiguities, leading to Q-LAMBDA abandoning the asymptotic efficiency earlier than MC-LAMBDA, and to the overall performance degradation. Within this region, fixed solutions can slightly perform worse than the float ones, with such phenomena being also observed for the real and integer mixed model in [8]. The threshold point, i.e., the noise level from which the estimator loses its asymptotic efficiency, depends on the geometry of the problem, multi-antenna setup and noise level. The prediction of this point for the attitude mixed model remains an open challenge.

c) Large noise regime ($\sigma_\rho \geq 10$ m): as σ_ρ increases, the observations carry too little information to resolve the integer ambiguities. Forcing a fix introduces biases, violating the bounds' unbiasedness assumptions. Thus, the estimators are no longer informative and their IMSE exceeds the CRB. For all estimators, fixed and float solutions coincide and providing no gain from integer constraints.

When comparing estimators: –LAMBDA is suboptimal because of not directly estimating the orientation. However, its appeal is a straightforward, linear baseline solution that is insensitive to initialization. –Direct methods (Q- and MC-LAMBDA) are optimal in the low-noise regime, with MC-LAMBDA being the overall best performer. This is due to the exhaustive evaluation of the cost function in (36) over all integer vector candidates. In contrast, Q-LAMBDA obtains the fixed attitude solution from the integer vector obtained from the ILS. In terms of computational resources, runtime scales as LAMBDA : Q-LAMBDA : MC-LAMBDA $\approx 1:2:14$.

VI. Conclusion

The main object of this article was the derivation of hybrid lower bounds on the estimation problem containing both Lie group and integer parameters. New closed-form expressions of the McAulay-Seidman bound (LG-IS-MSB) and Cramér-Rao bound (LG-IS-CRB) on the Lie group-integer space were provided, the LG-IS-CRB being the limiting form of the LG-IS-MSB. The general LG-IS-CRB was particularized for multi-antenna GNSS attitude estimation, where the unknowns are a rotation matrix in SO(3) and a set of integer carrier phase ambiguities. This application was used to validate the proposed bound through numerical simulations in a realistic scenario, showing i) that asymptotically efficient estimators exist for the Lie group-integer space regression model, and ii) that the proposed bound is able to predict the IMSE performance.

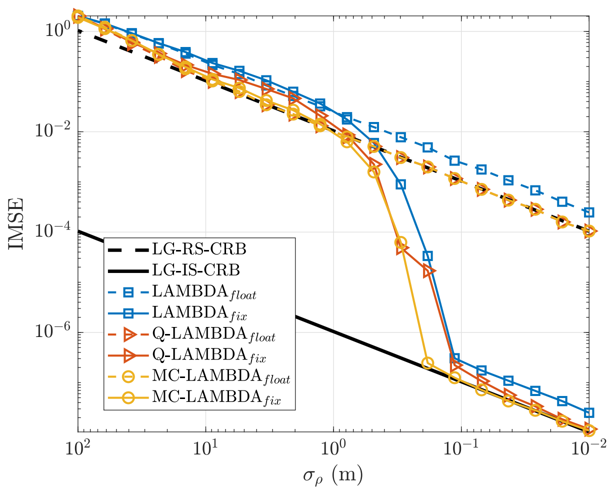

Fig. 2. IMSE for the rotation matrix $\mathbf{R}_0$ versus the undifferenced code measurements noise standard deviation σ_ρ ($\sigma_\Phi = \sigma_\rho/100$); Performance of *float* and *fix* estimates for LAMBDA, Q-LAMBDA and MC-LAMBDA; and the corresponding bounds: LG-RS-CRB (float) and LG-IS-CRB (fix).

Appendix A
Proofs of Theorems

A. Proof of Theorem 1

Similarly to (2), an unbiased estimator $\widehat{\mathbf{M}}_0$ on the space $G \times \mathbb{Z}^K$ can be built by taking advantage of the group logarithm $\boldsymbol{l}_{G\times\mathbb{Z}^K}(.)$ and respecting (9). This condition, being a continuum of constraints, can be discretized by defining a set of test points on $G\times\mathbb{Z}^K$, $\mathbf{M}^{(1:L)} = \left[\mathbf{M}_{LG}^{(1:L_1)}\ \mathbf{M}_{IS}^{(1:L-L_1)}\right]$, which admit the following structure:

$$\mathbf{M}_{LG}^{(l)} = \begin{bmatrix} \mathbf{R}_0^{(l)} & \mathbf{0}_{n\times K} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{K\times n} & \mathbf{I}_{n\times n} & \mathbf{0}_{K\times 1} \\ \mathbf{0}_{1\times n} & \mathbf{0}_{1\times K} & 1 \end{bmatrix}, \mathbf{M}_{IS}^{(l)} = \begin{bmatrix} \mathbf{I}_{n\times n} & \mathbf{0}_{n\times K} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{K\times n} & \mathbf{I}_{n\times n} & \mathbf{z}^{(l)} \\ \mathbf{0}_{1\times n} & 0_{1\times K} & 1 \end{bmatrix}.$$

Then, a discretization of (9) is given by:

$$\mathbb{E}\left(\mathbf{v}_{\mathbf{M}_0}\left(\mathbf{Y};\{\mathbf{M}^{(1:L)}\}\right)\text{Log}_{G\times\mathbb{Z}^K}\left(\mathbf{M}_0^{-1}\widehat{\mathbf{M}}_0\right)^\top\right) =$$

$$\left[\boldsymbol{l}_{G\times\mathbb{Z}^K}\left(\mathbf{M}_0^{-1}\mathbf{M}_{LG}^{(1)}\right),\cdots,\boldsymbol{l}_{G\times\mathbb{Z}^K}\left(\mathbf{M}_0^{-1}\mathbf{M}_{LG}^{(L_1)}\right),\right. \quad (39)$$

$$\left.\boldsymbol{l}_{G\times\mathbb{Z}^K}\left(\mathbf{M}_0^{-1}\mathbf{M}_{IS}^{(1)}\right)\cdots,\boldsymbol{l}_{G\times\mathbb{Z}^K}\left(\mathbf{M}_0^{-1}\mathbf{M}_{IS}^{(L-L_1)}\right)\right] \quad (40)$$

with $\left[\mathbf{v}_{\mathbf{M}_0}\left(\mathbf{Y};\{\mathbf{M}^{(1:L)}\}\right)\right]_l = \dfrac{p(\mathbf{Y}|\mathbf{M}_{LG}^{(l)})}{p(\mathbf{Y}|\mathbf{M}_0)} \forall l \in \{1,\dots,L_1\}$, and $\left[\mathbf{v}_{\mathbf{M}_0}\left(\mathbf{Y};\{\mathbf{M}^{(1:L)}\}\right)\right]_l = \dfrac{p(\mathbf{Y}|\mathbf{M}_{IS}^{(l-L_1)})}{p(\mathbf{Y}|\mathbf{M}_0)}$ $\forall l \in \{L_1+1,\dots,L\}$. By using [34, Lemma 1], it results the inequality:

$$\mathbb{E}\left(\boldsymbol{l}_{G\times\mathbb{Z}^K}\left(\mathbf{M}_0^{-1}\widehat{\mathbf{M}}_0\right)\boldsymbol{l}_{G\times\mathbb{Z}^K}\left(\mathbf{M}_0^{-1}\widehat{\mathbf{M}}_0\right)^\top\right) \succeq \boldsymbol{\Delta}_{\text{LG-IS}}\,\mathcal{I}_{\text{LG-IS-MSB}}^{-1}\boldsymbol{\Delta}_{\text{LG-IS}}^\top, \quad (41)$$

the left-hand term being the intrinsic MSE.

B. Proof of Theorem 2

Proof. Let us define the matrix $\mathbf{T} = \left[\mathbf{T}_M^\top, \mathbf{T}_z^\top\right]$, with

$$\mathbf{T}_M = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ -\frac{1}{\delta_1} & \frac{1}{\delta_1} & 0 & \dots & 0 \\ -\frac{1}{\delta_2} & 0 & \frac{1}{\delta_2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ -\frac{1}{\delta_p} & 0 & \dots & 0 & \frac{1}{\delta_p} \end{bmatrix}, \mathbf{T}_z = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ -1 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

Multiplying the equality (40) by $\mathbf{T}^\top$, and using (11)-(12), we obtain the new condition:

$$\mathbb{E}\left(\tilde{\mathbf{v}}_{\mathbf{M}_0}\left(\mathbf{Y};\{\mathbf{M}_0^{(1:L)}\}\right)\boldsymbol{l}_{G\times\mathbb{Z}^K}\left(\mathbf{M}_0^{-1}\widehat{\mathbf{M}}_0\right)^\top\right) =$$

$$= \begin{bmatrix} \mathbf{i}_1 & \cdots & \mathbf{i}_P & 0 & \cdots & 0 \\ \mathbf{0} & \cdots & \mathbf{0} & \left(\boldsymbol{\gamma}^{(1)}\right) & \cdots & \left(\boldsymbol{\gamma}^{(2K)}\right)^\top \end{bmatrix} \triangleq \tilde{\boldsymbol{\Delta}}^\top(\{\boldsymbol{\gamma}^{(i)}\}_{i=1}^{2K}), \quad (42)$$

with $\forall l \in \{1,\dots P\}$,

$$\left[\tilde{\mathbf{v}}_{\mathbf{M}_0}(\mathbf{Y};\{\mathbf{M}^{(1:L)}\}\right]_l = \frac{p\left(\mathbf{Y}|\mathbf{M}_{LG}^{(l)}\right) - p(\mathbf{Y}|\mathbf{M}_0)}{\delta_p\, p(\mathbf{Y}|\mathbf{M}_0)},$$

and $\forall l \in \{P+1,\dots,L\}$,

$$\left[\tilde{\mathbf{v}}_{\mathbf{M}_0}(\mathbf{Y};\{\mathbf{M}^{(1:L)}\}\right]_l = \frac{p\left(\mathbf{Y}|\mathbf{M}_{IS}^{(l)}\right) - p(\mathbf{Y}|\mathbf{M}_0)}{p(\mathbf{Y}|\mathbf{M}_0)}.$$

First, (42) allows us to provide the following matrix inequality:

$$\mathbb{E}\left(\text{Log}_{G\times\mathbb{Z}^K}^\vee\left(\mathbf{M}_0^{-1}\widehat{\mathbf{M}}_0\right)\ \text{Log}_{G\times\mathbb{Z}^K}^\vee\left(\mathbf{M}_0^{-1}\widehat{\mathbf{M}}_0\right)^\top\right) \succeq \tilde{\boldsymbol{\Delta}}(\{\boldsymbol{\gamma}^{(i)}\}_{i=1}^{2K})\ \underbrace{\tilde{\mathcal{I}}_{\text{LG-IS-MSB}}^{-1}(\{\delta_p\}_{p=1}^P,\{\boldsymbol{\gamma}^{(i)}\}_{i=1}^{2K})}_{=\mathbb{E}\left(\tilde{\mathbf{v}}_{\mathbf{M}_0}(\mathbf{Y};\{\mathbf{M}^{(1:L)}\})\,\tilde{\mathbf{v}}_{\mathbf{M}_0}(\mathbf{Y};\{\mathbf{M}^{(1:L)}\})^\top\right)}\ \tilde{\boldsymbol{\Delta}}(\{\boldsymbol{\gamma}^{(i)}\}_{i=1}^{2K})^\top$$

The right-hand term can be interpreted as new expression of the hybrid MSB on $G\times\mathbb{Z}^K$ depending on the test points $\mathbf{M}^{(1:L)}$, which in turn depend on LG increments $\{\delta_p\}_{p=1}^P$ and integer increment $\{\boldsymbol{\gamma}^{(i)}\}_{i=1}^{2K}$. Consequently, by an analog reasoning with the mixed Euclidean CRB proposed in [8], the LG-IS-CRB can be obtained by tending towards 0 the LG test points increments δ_p:

$$\mathbf{P}_{\text{LG-IS-CRB}}(\{\boldsymbol{\gamma}^{(i)}\}_{i=1}^{2K}) = \tilde{\boldsymbol{\Delta}}(\{\boldsymbol{\gamma}^{(i)}\}_{i=1}^{2K})\ \underbrace{\lim_{\{\delta_p\}_{p=1}^P\mapsto\mathbf{0}}\tilde{\mathcal{I}}_{\text{LG-IS-MSB}}^{-1}(\{\delta_p\}_{p=1}^P,\{\boldsymbol{\gamma}^{(i)}\}_{i=1}^{2K})}_{\mathcal{I}_{\text{LG-IS-CRB}}^{-1}}\ \tilde{\boldsymbol{\Delta}}(\{\boldsymbol{\gamma}^{(i)}\}_{i=1}^{2K})^\top. \quad (43)$$

As $\mathbf{M}_0$ decomposes into two sub-parameters, $\mathbf{R}_0 \in G$ and $\mathbf{z}_0 \in \mathbb{Z}^K$, $\tilde{\mathcal{I}}_{\text{LG-IS-MSB}}$ is written as a 2×2 block matrix:

$$\begin{aligned}\left[\tilde{\mathcal{I}}_{\text{LG-IS-MSB}}\right]_{1,1} &= \mathcal{I}_{\mathbf{R}_0}(\{\delta_p\}_{p=1}^P) \\ &= \mathbb{E}\left(\mathbf{c}_1(\{\delta_p\}_{p=1}^P)\,\mathbf{c}_1\left(\{\delta_p\}_{p=1}^P\right)^\top\right), \end{aligned} \tag{44}$$

$$\begin{aligned}\left[\tilde{\mathcal{I}}_{\text{LG-IS-MSB}}\right]_{2,2} &= \mathcal{I}_{\mathbf{z}_0}(\{\boldsymbol{\gamma}^{(i)}\}_{i=1}^{2K}) \\ &= \mathbb{E}\left(\mathbf{c}_2(\{\boldsymbol{\gamma}^{(i)}\}_{i=1}^{2K})\,\mathbf{c}_2\left(\{\boldsymbol{\gamma}^{(i)}\}_{i=1}^{2K}\right)^\top\right), \end{aligned} \tag{45}$$

$$\begin{aligned}\left[\tilde{\mathcal{I}}_{\text{LG-IS-MSB}}\right]_{1,2} &= \mathcal{I}_{\mathbf{R}_0,\mathbf{z}_0}(\{\delta_p\}_{p=1}^P, \{\boldsymbol{\gamma}^{(i)}\}_{i=1}^{2K}) \\ &= \mathbb{E}\left(\mathbf{c}_1(\{\delta_p\}_{p=1}^P)\,\mathbf{c}_2\left(\{\boldsymbol{\gamma}^{(i)}\}_{i=1}^{2K}\right)^\top\right), \end{aligned} \tag{46}$$

with $\forall\, p \in [\![1:P]\!]$, $\forall\, j \in [\![1:2K]\!]$:

$$\left[\mathbf{c}_1(\{\delta_p\}_{p=1}^P)\right]_p = \frac{\tilde{p}\left(\mathbf{Y};\mathbf{M}_0, \mathbf{i}_p\,\delta_p\right) - p(\mathbf{Y}|\mathbf{M}_0)}{\delta_p\, p(\mathbf{Y}|\mathbf{M}_0)}, \tag{47}$$

$$\left[\mathbf{c}_2(\{\boldsymbol{\gamma}^{(i)}\}_{i=1}^{2K})\right]_j = \frac{\tilde{p}\left(\mathbf{Y};\mathbf{M}_0, \boldsymbol{\gamma}^{(j)}\right) - p(\mathbf{Y}|\mathbf{M}_0)}{p(\mathbf{Y}|\mathbf{M}_0)}. \tag{48}$$

Then, the $\mathcal{I}_{\text{LG-IS-CRB}}$ of interest is given by,

$$\mathcal{I}_{\text{LG-IS-CRB}} = \begin{bmatrix} \lim\limits_{\{\delta_p\}_{p=1}^P \mapsto \mathbf{0}} \mathcal{I}_{\mathbf{R}_0}(\{\delta_p\}_{p=1}^P) & \lim\limits_{\{\delta_p\}_{p=1}^P \mapsto \mathbf{0}} \mathcal{I}_{\mathbf{R}_0,\mathbf{z}_0}(\{\delta_p\}_{p=1}^P) \\ \lim\limits_{\{\delta_p\}_{p=1}^P \mapsto \mathbf{0}} \mathcal{I}_{\mathbf{R}_0,\mathbf{z}_0}(\{\delta_p\}_{p=1}^P)^\top & \mathcal{I}_{\mathbf{z}_0} \end{bmatrix} \tag{49}$$

- *Expression of* $\lim\limits_{\{\delta_p\}_{p=1}^P \mapsto \mathbf{0}} \mathcal{I}_{\mathbf{R}_0}(\{\delta_p\}_{p=1}^P)$:

By assuming that $\mathbf{c}_1\left(\{\delta_p\}_{p=1}^P\right)$ is upper-bounded by a continuous integrable function, then $\boldsymbol{\delta} \mapsto \mathbb{E}\left(\mathbf{c}_1\left(\{\delta_p\}_{p=1}^P\right)\mathbf{c}_1\left(\{\delta_p\}_{p=1}^P\right)^\top\right)$ is continuous and

$$\begin{aligned} &\lim_{\{\delta_p\}_{p=1}^P \mapsto \mathbf{0}} \mathcal{I}_{\mathbf{R}_0}(\{\delta_p\}_{p=1}^P) = \\ &\mathbb{E}\left(\lim_{\{\delta_p\}_{p=1}^P \mapsto \mathbf{0}} \mathbf{c}_1(\{\delta_p\}_{p=1}^P)\,\mathbf{c}_1(\{\delta_p\}_{p=1}^P)^\top\right) = \\ &\mathbb{E}\left(\frac{\partial \log \tilde{p}(\mathbf{Y};\mathbf{M}_0,\boldsymbol{\epsilon})}{\partial \boldsymbol{\epsilon}} \frac{\partial \log \tilde{p}(\mathbf{Y};\mathbf{M}_0,\boldsymbol{\epsilon})}{\partial \boldsymbol{\epsilon}}^\top \Bigg|_{\boldsymbol{\epsilon}=\mathbf{0}}\right), \end{aligned} \tag{50}$$

that proves equation (16).

- *Expression of* $\lim\limits_{\{\delta_p\}_{p=1}^P \mapsto \mathbf{0}} \mathcal{I}_{\mathbf{R}_0,\mathbf{z}_0}(\{\delta_p\}_{p=1}^P, \{\boldsymbol{\gamma}^{(i)}\}_{i=1}^{2K})$:

$$\begin{aligned} &\lim_{\{\delta_p\}_{p=1}^P \mapsto \mathbf{0}} \mathcal{I}_{\mathbf{R}_0,\mathbf{z}_0}(\{\delta_p\}_{p=1}^P, \{\boldsymbol{\gamma}^{(i)}\}_{i=1}^{2K}) = \\ &\mathbb{E}\left(\left(\lim_{\{\delta_p\}_{p=1}^P \mapsto \mathbf{0}} \mathbf{c}_1(\{\delta_p\}_{p=1}^P)\right)\mathbf{c}_2(\{\boldsymbol{\gamma}^{(i)}\}_{i=1}^{2K})^\top\right) \\ &= \mathbb{E}\left(\frac{\partial \log \tilde{p}(\mathbf{Y};\mathbf{M},\boldsymbol{\epsilon})}{\partial \boldsymbol{\epsilon}}\Bigg|_{\boldsymbol{\epsilon}=\mathbf{0}} \mathbf{c}_2(\{\boldsymbol{\gamma}^{(i)}\}_{i=1}^{2K})^\top\right), \end{aligned} \tag{51}$$

that proves equation (17).

- *Expression of* $\mathcal{I}_{\mathbf{z}_0}$:

By injecting (48) in (45), it is straightforward that,

$$\begin{aligned} (\mathcal{I}_{\mathbf{z}_0})_{i,j} &= \mathbb{E}\left(\frac{\tilde{p}(\mathbf{Y}|\mathbf{M},\boldsymbol{\gamma}^{(i)})}{p(\mathbf{Y}|\mathbf{M})}\frac{\tilde{p}(\mathbf{Y}|\mathbf{M},\boldsymbol{\gamma}^{(j)})}{p(\mathbf{Y}|\mathbf{M})}\right) - \\ &\underbrace{\mathbb{E}\left(\frac{\tilde{p}(\mathbf{Y}|\mathbf{M},\boldsymbol{\gamma}^{(j)})}{p(\mathbf{Y}|\mathbf{M})}\right)}_{=1} - \underbrace{\mathbb{E}\left(\frac{\tilde{p}(\mathbf{Y}|\mathbf{M},\boldsymbol{\gamma}^{(i)})}{p(\mathbf{Y}|\mathbf{M})}\right)}_{=1} + 1 \\ &= \mathbb{E}\left(\frac{\tilde{p}(\mathbf{Y}|\mathbf{M},\boldsymbol{\gamma}^{(i)})}{p(\mathbf{Y}|\mathbf{M})}\frac{\tilde{p}(\mathbf{Y}|\mathbf{M},\boldsymbol{\gamma}^{(j)})}{p(\mathbf{Y}|\mathbf{M})}\right) - 1, \end{aligned} \tag{52}$$

that proves equation (18). Then, by substituting (50) and (51) in (49), we obtain (13). □

C. Proof of Theorem 3

- *Computation of* $\mathcal{I}_{\mathbf{R}_0}$:

First, using the LG Slepian-Bangs formula derived in [21]:

$$\begin{aligned} &\forall (i,j) \in [\![1:3]\!]^2, (\mathcal{I}_{\mathbf{R}_0})_{i,j} = \\ &\mathbb{E}\left(\frac{\partial \log \tilde{p}(\mathbf{y};\mathbf{R}_0,\boldsymbol{\epsilon})}{\partial \epsilon_i}\frac{\partial \log \tilde{p}(\mathbf{y};\mathbf{R}_0,\boldsymbol{\epsilon})}{\partial \epsilon_j}^\top\right)\Bigg|_{\boldsymbol{\epsilon}=\mathbf{0}} \\ &= \frac{\partial \mathbf{h}(\mathbf{R}_0 \mathrm{Exp}^\wedge_{SO(3)}\left(\boldsymbol{\epsilon}\right)\ ,\mathbf{z}_0)}{\partial \epsilon_i}^\top \\ &\times \boldsymbol{\Sigma}^{-1}\frac{\partial \mathbf{h}(\mathbf{R}_0\, \mathrm{Exp}^\wedge_{SO(3)}\left(\boldsymbol{\epsilon}\right)\ ,\mathbf{z}_0)}{\partial \epsilon_j}\Bigg|_{\boldsymbol{\epsilon}=\mathbf{0}}, \forall \boldsymbol{\epsilon} \in \mathbb{R}^3, \\ &\frac{\partial \mathbf{h}(\mathbf{R}_0 \mathrm{Exp}^\wedge_{SO(3)}\left(\boldsymbol{\epsilon}\right)\ ,\mathbf{z}_0)}{\partial \epsilon_i} = \\ &\left(\begin{bmatrix}\mathbf{I}_N \\ \mathbf{I}_N\end{bmatrix} \otimes \left(\mathbf{C}\mathbf{R}_0 \frac{\partial \mathrm{Exp}^\wedge_{SO(3)}\left(\boldsymbol{\epsilon}\right)}{\partial \epsilon_i}\right)\right)\mathbf{b}. \end{aligned}$$

As $\frac{\partial \mathrm{Exp}^\wedge_{SO(3)}\left(\boldsymbol{\epsilon}\right)}{\partial \epsilon_i}\Bigg|_{\boldsymbol{\epsilon}=\mathbf{0}} = \mathbf{G}_i, \forall i \in [\![1:3]\!]$, with $\mathbf{G}_i$ being the ith-basis of $\mathfrak{so}(3)$, $\mathbf{G}_1 = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -1 \\ 0 & 1 & 0\end{bmatrix}$, $\mathbf{G}_2 = \begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0 \\ -1 & 0 & 0\end{bmatrix}$, and $\mathbf{G}_3 = \begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}$, then $\mathcal{I}_{\mathbf{R}_0}$ in (30) is proven.

- *Computation of* $\mathcal{I}_{\mathbf{R}_0,\mathbf{z}_0}$:

We remind that,

$$\begin{aligned} (\mathcal{I}_{\mathbf{R}_0,\mathbf{z}_0})_{i,k} &= \mathbb{E}\left(\frac{\partial \log p(\mathbf{y}|\mathbf{M}_0\, \mathrm{Exp}^\wedge_{SO(3)\times\mathbb{Z}^K}\left([\boldsymbol{\epsilon},\mathbf{0}]^\top\right)}{\partial \epsilon_i}\Bigg|_{\boldsymbol{\epsilon}=\mathbf{0}}\right. \\ &\times \left.\left(\frac{p(\mathbf{y}|\mathbf{M}_0\, \mathrm{Exp}^\wedge_{SO(3)\times\mathbb{Z}^K}\left([\mathbf{0},\boldsymbol{\gamma}^{(k)}]^\top\right)}{p(\mathbf{y}|\mathbf{M}_0)} - 1\right)\right). \end{aligned} \tag{53}$$

In [8], Appendix B, it is demonstrated that if $\mathbf{y}$ follows a Gaussian distribution with mean $\mathbf{m}(\boldsymbol{\theta}) + \mathbf{E}\,\mathbf{z}$ ($\boldsymbol{\theta} \in \mathbb{R}^p, \mathbf{z} \in \mathbb{Z}^K$) and covariance $\mathbf{S}$, i.e.,

$$\mathbf{y} \sim \mathcal{N}(\mathbf{m}(\boldsymbol{\theta}) + \mathbf{E}\,\mathbf{z}, \mathbf{S}), \tag{54}$$

and if a set of integer test points $\{\mathbf{z}^{(k)}\}_{k=1}^{2K}$ are available, then

$$\mathbb{E}\left(\frac{\partial \log p(\mathbf{y}|\boldsymbol{\theta},\mathbf{z})}{\partial \boldsymbol{\theta}}\left(\frac{p(\mathbf{y}|\boldsymbol{\theta},\mathbf{z}^{(k)})}{p(\mathbf{y}|\boldsymbol{\theta},\mathbf{z})} - 1\right)\right) = \frac{\partial \mathbf{m}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}^\top \mathbf{S}^{-1}\mathbf{E}\left(\mathbf{z}^{(k)} - \mathbf{z}\right),$$

(with assumption that $\mathbb{E}\left(\frac{\partial \log p(\mathbf{y}|\boldsymbol{\theta},\mathbf{z})}{\partial \boldsymbol{\theta}}\right) = \mathbf{0}$). Then, using this result for the Gaussian model (26) with mean $\mathbf{h}(\mathbf{R}_0, \mathbf{z}_0)$ and covariance $\boldsymbol{\Sigma}$ yields,

$$(\mathcal{I}_{\mathbf{R}_0,\mathbf{z}_0})_{i,k} = \left.\frac{\partial \mathbf{h}(\mathbf{R}_0 \operatorname{Exp}^{\wedge}_{SO(3)\times\mathbb{Z}^K}([\boldsymbol{\epsilon},\mathbf{0}]) \;, \mathbf{z}_0)}{\partial \epsilon_i}\right|^{\top}_{\boldsymbol{\epsilon}=\mathbf{0}} \times \boldsymbol{\Sigma}^{-1} \mathbf{A} \underbrace{\left(\mathbf{z}^{(k)} - \mathbf{z}_0\right)}_{\boldsymbol{\gamma}^{(k)}}, \tag{55}$$

$$\frac{\partial \mathbf{h}(\mathbf{R}_0 \operatorname{Exp}^{\wedge}_{SO(3)\times\mathbb{Z}^K}([\boldsymbol{\epsilon},\mathbf{0}]) \;, \mathbf{z}_0)}{\partial \epsilon_i} = \left(\begin{bmatrix}\mathbf{I}_N\\ \mathbf{I}_N\end{bmatrix} \otimes (\mathbf{C}\mathbf{R}_0\mathbf{G}_i)\right)\mathbf{b},$$

then $\mathcal{I}_{\mathbf{R}_0,\mathbf{z}_0}$ in (31) is proven.

- *Computation of* $\mathcal{I}_{\mathbf{z}_0}$:

We remind that $\mathcal{I}_{\mathbf{z}_0} = \mathbb{E}\left(\mathbf{c}\,\mathbf{c}^{\top}\right) - \mathbf{1}_{2K}\,\mathbf{1}_{2K}^{\top}$. Again using [8], the model (54) provides that,

$$\mathbb{E}\left(\frac{p(\mathbf{y}|\boldsymbol{\theta},\mathbf{z}^{(l)})}{p(\mathbf{y}|\boldsymbol{\theta},\mathbf{z})}\,\frac{p(\mathbf{y}|\boldsymbol{\theta},\mathbf{z}^{(k)})}{p(\mathbf{y}|\boldsymbol{\theta},\mathbf{z})}\right) = \exp\left(0.5\left(\tilde{\mathbf{m}}_{l,k}^{\top}\,\mathbf{S}\tilde{\mathbf{m}}_{l,k} - \tilde{\delta}_{l,k}\right)\right),$$

$$\tilde{\mathbf{m}}_{l,k} = \mathbf{S}^{-1}\left(\mathbf{m}(\boldsymbol{\theta}) + \mathbf{E}\,\mathbf{z}^{(l)}\right) + \mathbf{S}^{-1} \times \left(\mathbf{m}(\boldsymbol{\theta}) + \mathbf{E}\mathbf{z}^{(k)}\right) - \mathbf{S}^{-1}\left(\mathbf{m}(\boldsymbol{\theta}) + \mathbf{E}\,\mathbf{z}\right),$$

$$\tilde{\delta}_{l,k} = \left(\mathbf{m}(\boldsymbol{\theta}) + \mathbf{E}\,\mathbf{z}^{(l)}\right)^{\top}\mathbf{S}^{-1}\left(\mathbf{m}(\boldsymbol{\theta}) + \mathbf{E}\mathbf{z}^{(l)}\right) + \left(\mathbf{m}(\boldsymbol{\theta}) + \mathbf{E}\mathbf{z}^{(k)}\right)^{\top}\mathbf{S}^{-1}\left(\mathbf{m}(\boldsymbol{\theta}) + \mathbf{E}\,\mathbf{z}^{(k)}\right) - \left(\mathbf{m}(\boldsymbol{\theta}) + \mathbf{E}\,\mathbf{z}\right)^{\top}\mathbf{S}^{-1}\left(\mathbf{m}(\boldsymbol{\theta}) + \mathbf{E}\mathbf{z}\right).$$

Using these expressions we obtain (33) and (34).

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